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DEFORMATION OF INFINITE DIMENSIONAL DIFFERENTIAL GRADED LIE ALGEBRAS

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Abstract. For a wide class of topological infinite dimensional differential graded Lie algebras the complete set of deformations is constructed. It is shown that for several deformation problems the existence of a formal power series solution guarantees the existence of an analytic solution.

0. Introduction

0.1. In [NR2] Nijenhuis and Richardson developed deformation theory of finite dimensional differential graded Lie algebras. In particular, for a differential graded Lie algebra (L, d) they constructed a versal deformation, which is a precise analogue of the versal deformation of a compact complex manifold constructed by Kuranishi ([Ku1], [Ku2]). The parameter space of this deformation is a germ of a finite dimensional analytic variety. This germ is called *the Kuranishi space* of L.

0.2. In [GM2] Goldman and Millson defined a notion of *analytic differential graded Lie algebra*. Namely, a differential graded Lie algebra $(L = \bigoplus L^i, d)$ is analytic if

1. it has finite cohomology in degrees 0 and 1,
2. each L^i is a normed vector space,
3. the maps $d : L^i \to L^{i+1}$, and $\operatorname{ad} a : L^i \to L^{i+1}$ $(a \in L^1)$ are continuous,
4. the completion $\widehat{L}^i$ of L^i may be decomposed into a direct sum of closed subspaces
$$\widehat{L}^i = \widehat{B}^i \oplus \mathcal{H}^i \oplus \widehat{A}^i,$$
where $\widehat{B}^i$ is the image of d in $\widehat{L}^i$, $\mathcal{H}^i + \widehat{B}^i$ is the kernel of d in $\widehat{L}^i$ and $\mathcal{H}^i \subset L^i$.

For an analytic differential graded Lie algebra Goldman and Millson constructed the Kuranishi space, which is a germ of a finite dimensional analytic variety, as in the case of finite dimensional differential graded Lie algebras. It is shown in [GM2] that the Kuranishi space is invariant under quasi-isomorphisms of differential graded Lie algebras.

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0.3. For a general analytic differential graded Lie algebra it is impossible to define the equivalence of deformations and the Kuranishi space may be considered as a complete set of deformations only formally.

In this paper we define an *elliptic differential graded Lie algebra* to be an analytic differential graded Lie algebra such that for any element $a\in\widehat{L}^0$ the operator $\operatorname{ad} a:\widehat{L}^1\to\widehat{L}^1$ generates a one parameter semigroup of bounded operators $\exp(t\operatorname{ad} a):\widehat{L}^1\to\widehat{L}^1$. The class of elliptic differential graded Lie algebras contains such examples as the finite dimensional differential graded Lie algebra, the twisted de Rham algebra of differential forms with values in endomorphisms of a flat vector bundle over a compact manifold, etc.

0.4. By an (n - parameter) *analytic deformation* of a differential graded Lie algebra (L,d) we shall mean (cf. [NR2]) an analytic map

$$m:U\to L^1,\qquad u\mapsto\ m(u)$$

defined on a neighborhood U of 0 in $\mathbb{R}^n$, such that $m(0)=0$ and

$$dm(u)+\frac{1}{2}[m(u),m(u)]=0.$$

For each $a\in\widehat{L}^0$ we define the operator $\rho(a):\widehat{L}^1\to\widehat{L}^1$ by the formula

$$\rho(a):x\mapsto\exp(\operatorname{ad} a)\,x-\left[\int_0^1\exp((1-t)\operatorname{ad} a)\,dt\right]\,da.$$

0.5. We shall say that two analytic deformations $m(u)$ $(u\in U\subset\mathbb{R}^n)$ and $m'(u)$ $(u\in U'\subset\mathbb{R}^n)$ are *equivalent* if there exists a neighborhood of zero $V\subset U\cap U'$ such that for any $v\in V$ there exists $f\in\widehat{L}^0$ such that $\rho(f)\,m(v)=m'(v)$. If the element f may be chosen so that $f\in\widehat{A}^0$ we shall say that the deformations $m(u)$ and $m'(u)$ are *similar*.

0.6. An analytic deformation $m:U\to L^1$ $(0\in U\subset\mathbb{R}^n)$ will be called a *Kuranishi deformation* if $m(u)\in\mathcal{H}^1\oplus\widehat{A}^1$ provided u is small enough.

The following proposition generalizes a result of Nijenhuis and Richardson ([NR2]) :

Proposition 1. *Any analytic deformation of an elliptic differential graded Lie algebra is similar to a unique Kuranishi deformation.*

0.7. We use the above result to show that for some deformation problems the existence of a formal power series solution guarantees the existence of an analytic solution. More precise, we define a *formal deformation* of differential graded Lie algebra (L,d) to be a formal power series

$$m(t)=tm_1+t^2m_2+\cdots$$

with coefficients m_n $(n=1,2,\dots)$ in L^1 satisfying

$$dm(t)+\frac{1}{2}[m(t),m(t)]=0.$$

(For simplicity we consider only 1-parameter formal deformations. Generalizations to the n-parameter case are immediate). Two formal deformations $m(t)$ and $m'(t)$ will be called *formally similar* if there exists a formal power series

$$f(t) = tf_1 + t^2 f_2 + \cdots$$

with coefficients f_n $(n = 1, 2, \dots)$ in $\widehat{A}^0 \subset \widehat{L}^0$ such that

$$m'(t) = \rho(f(t))\, m(t) \overset{\text{def}}{=} \exp(\operatorname{ad} f(t))\, m(t) + \frac{I - \exp(\operatorname{ad} f(t))}{\operatorname{ad} f(t)}(df(t)).$$

Our main results are the following two theorems:

Theorem 2. *If two analytic deformations of an elliptic differential graded Lie algebra are formally similar, then they are similar.*

Theorem 3. *For any formal deformation $m(t)$ of an elliptic differential graded Lie algebra (L, d) and any integer n there exists an analytic deformation $\widetilde{m}(t)$ such that*

$$\widetilde{m}(t) \equiv m(t) \qquad \operatorname{mod}\ t^n.$$

Note that in finite dimensional case the above theorems follow directly from Artin's result ([A]).

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1. Elliptic Differential Graded Lie Algebras

In this section we define the notion of elliptic differential graded Lie algebra, which will be the main object of study in the present paper.

1.1. A *graded Lie algebra* ([NR2], [GM1]) is a complex vector space $L = \bigoplus_{0 \le i \le N} L^i$ graded by a nonnegative integers, and a family of bilinear maps

$$[\cdot, \cdot] : L^i \times L^j \to L^{i+j}$$

satisfying (graded) skew-commutativity:

$$[a, b] + (-1)^{ij}[b, a] = 0$$

and the (graded) Jacobi identity:

$$[a, [b, c]] = [[a, b], c] + (-1)^{ij}[b, [a, c]]$$

where $a \in L^i,\ b \in L^j,\ c \in L^k$.

1.2. A *differential graded Lie algebra* is a pair (L, d) where L is a graded Lie algebra and d is a family of linear maps $d : L^i \to L^{i+1}$ $(0 \le i \le N)$ such that $d \circ d = 0$ and

$$d[a,b] = [da,b] + (-1)^i [a, db]$$

where $a \in L^i,\ b \in L$.

1.3. A *normed differential graded Lie algebra* ([GM2]) is a differential graded Lie algebra (L,d) equipped with a norm $|\cdot|_i$ on each L^i making L^i into a normed vector space such that the maps

$$d : L^i \to L^{i+1}$$

and

$$\operatorname{ad} a : L^i \to L^{i+1}, \qquad a \in L^1$$

are continuous.

Note that in [GM2] the map $\operatorname{ad} a : L^i \to L^{i+1}$ $\ a \in L^0$ is demanded to be continuous only for $i = 1$.

Let $\widehat{L}^i$ be the completion of L^i for $i = 0, 1, 2, \dots$. The vector space L^i is the analogue of the smooth i-forms and $\widehat{L}^i$ of the Sobolev i-forms.

1.4. An *analytic differential graded Lie algebra* ([GM2]) is a normed differential graded Lie algebra with finite dimensional cohomology in degree 0 and 1, equipped with decompositions of each $\widehat{L}^i$ into direct sums of closed subspaces:

$$\widehat{L}^i = \widehat{B}^i \oplus \mathcal{H}^i \oplus \widehat{A}^i, \tag{1.1}$$

where $\widehat{B}^i$ is the image of d in $\widehat{L}^i$, $\mathcal{H}^i + \widehat{B}^i$ is the kernel of d in $\widehat{L}^i$ and $\mathcal{H}^i \subset L^i$. Let $\beta : \widehat{L}^i \to \widehat{B}^i$, $\alpha : \widehat{L}^i \to \widehat{A}^i$ and $H : \widehat{L}^i \to \mathcal{H}^i$ be the corresponding projections.

We assume that all three projections carry L^i into itself and that $\beta(L^i) = dL^{i-1}$. Set $A^i = \widehat{A}^i \cap L^i$ and $B^i = \widehat{B}^i \cap L^i$. Obviously, $A^i = \alpha(L^i)$, $B^i = \beta(L^i)$ and there is an algebraic direct sum decomposition

$$L^i = B^i + \mathcal{H}^i + A^i. \tag{1.2}$$

There exists ([GM2]) a continuous operator $\delta : \widehat{L}^{i+1} \to \widehat{L}^i$ such that $\delta(L^{i+1}) \subset L^i$ $\widehat{A}^i = \operatorname{Im} \delta$ and

$$id = H + d \circ \delta + \delta \circ d. \tag{1.3}$$

1.5. We remind ([Ka] Ch. IX) that a family of bounded linear operators $U(t) : \widehat{L}^1 \to \widehat{L}^1$ depending on a parameter t $(0 \le t < \infty)$ is called a quasi-bounded one-parameter semigroup if $U(0) = id$,

$$U(t_1 + t_2) = U(t_1)U(t_2) \qquad (0 \le t_1, t_2 < \infty),$$

and there exist numbers $C > 0$ and β such that

$$\|U(t)\| \le Ce^{\beta t} \qquad (0 \le t < \infty). \tag{1.4}$$

In this case the strong limit

$$T = \operatorname*{s-lim}_{t \to +0} \frac{U(t) - id}{t}$$

exists and is a densely defined operator $\widehat{L}^1 \to \widehat{L}^1$. We shall say that T generates the semigroup $U(t)$ and write $U(t) = \exp(tT)$.

The description of the class of operators which generate a quasi-bounded one-parameter semigroup may be found in [Ka] Ch. IX §1.4. Note that any bounded operator belongs to this class.

1.6. Definition. By an *elliptic differential graded Lie algebra* we shall understand an analytic differential graded Lie algebra (L, d) such that for each $a \in \widehat{L}^0$ the operator $\mathrm{ad}\, a$ generates a quasi-bounded one-parameter semigroup of operators $\exp(t\, \mathrm{ad}\, a) : \widehat{L}^1 \to \widehat{L}^1$ depending continuously on $a \in \widehat{L}^0$, i.e. for any $\varepsilon > 0$ there exists a neighborhood U of 0 in $\widehat{L}^0$ such that for any $a \in U$ and for any $0 \le t \le 1$

$$\| \exp(t\, \mathrm{ad}\, a) - id\| < \varepsilon. \tag{1.5}$$

Here $\|\cdot\|$ denotes the standard norm on the space of operators acting on the Banach space $\widehat{L}^1$.

1.7. Remark. In many examples (cf. section 1.8) the operators $\mathrm{ad}\, a : \widehat{L}^1 \to \widehat{L}^1$ $(a \in \widehat{L}^0)$ are bounded. In this case $\exp(t\, \mathrm{ad}\, a)$ is given by a convergent power series

$$\exp(t\, \mathrm{ad}\, a) = \sum_{i=0}^{\infty} \frac{t^i}{i!} (\mathrm{ad}\, a)^i.$$

Hence any analytic differential graded Lie algebra of this type is elliptic. In particular, any finite dimensional differential graded Lie algebra is elliptic.

We finish this section with an example of an infinite dimensional differential graded Lie algebra.

1.8. The twisted de Rham algebra. Let M be a compact n-dimensional manifold and $\mathcal{E}$ a flat vector bundle over M. The vector bundle $End(\mathcal{E})$ inherits the flat structure from $\mathcal{E}$. The *twisted de Rham algebra* is the differential graded Lie algebra

$$(L, d) = \Big(\bigoplus_{i=0}^{n} A^i(M, End(\mathcal{E})), d \Big)$$

of all forms on M with values in the bundle $End(\mathcal{E})$.

Fix $s \in \mathbb{R}$ and equip the space $A^i(M, \mathrm{End}\,(\mathcal{E}))$ $(0 \le q \le n)$ with $(s - q)$-th Sobolev's norm. Then, by the classical Hodge theory, the algebra (L, d) is analytic. Since the map

$$A^0(M, End(\mathcal{E})) \times A^1(M, End(\mathcal{E})) \to A^1(M, End(\mathcal{E})),$$
$$(a, x) \mapsto [a, x]$$

is continuous this algebra is elliptic (cf. Remark 1.7).

2. Deformations of Elliptic Differential Graded Lie Algebras

In this section we define the notions of analytic deformations of elliptic differential graded Lie algebras and of analytic equivalence of two analytic deformations.

2.1. Let (L, d) be an elliptic differential graded Lie algebra. The equation

$$da + \frac{1}{2}[a, a] = 0, \qquad a \in L^1 \tag{2.1}$$

is called a *deformation equation* ([NR2]). We denote by $M \subset L^1$ the set of solutions of (2.1). For any $m \in M$ the derivation $d_m = d + \mathrm{ad}\, m : L \to L$ satisfies the

By an (n - parameter) *analytic deformation* of a differential graded Lie algebra (L, d) we shall mean an analytic map

$$m : U \to M, \qquad u \mapsto m(u)$$

defined on a neighborhood U of 0 in $\mathbb{R}^n$ such that $m(0) = 0$. The trivial map $m(u) \equiv 0$ defines the *trivial deformation*.

2.2. Fix $a \in \widehat{L}^0$, $x_0 \in \widehat{L}^1$ and let $x(t)$ denote the solution of the inhomogeneous differential equation

$$\frac{dx}{dt} = [a, x] - da, \qquad x(0) = x_0. \tag{2.2}$$

Obviously,

$$d + \operatorname{ad} x(t) = \exp(t \operatorname{ad} a) \circ \big(d + \operatorname{ad} x(0)\big) \circ \exp(-t \operatorname{ad} a).$$

We shall denote by $\rho(a)$ the map $\rho(a) : x_0 \mapsto x(1)$. By [Ka] Ch. IX §1.5 $\rho(a)$ is an affine transformation of $\widehat{L}^1$ given by the formula

$$\rho(a) : x_0 \mapsto \exp(\operatorname{ad} a)\, x_0 - \left[\int_0^1 \exp((1-t) \operatorname{ad} a)\, dt\right] da. \tag{2.3}$$

Formally one can rewrite (2.3) as (cf. [GM1] §1)

$$\rho(ta)\, x = \exp(t \operatorname{ad} a)\, x + \frac{I - \exp(t \operatorname{ad} a)}{\operatorname{ad} a}(da). \tag{2.4}$$

2.3. Let $\mathcal{G}$ denote the group of affine transformations of $\widehat{L}^1$ generated by the operators $\rho(a)$ with ($a \in \widehat{L}^0$).

Two elements $m, m' \in M$ are called *equivalent* if there exists $g \in \mathcal{G}$ such that $g(m) = m'$. If the element g may be chosen so that $g = \rho(f)$, $f \in \widehat{A}^0$ then the elements m and m' are called *similar*. Note that the similarity is not an equivalence relation.

Two analytic deformations $m(u)$ ($u \in U \subset \mathbb{R}^n$) and $m'(u)$ ($u \in U' \subset \mathbb{R}^n$) are called *equivalent* (respectively, *similar*) if there exists a neighborhood of zero $V \subset U \cap U'$ such that for any $v \in V$ the elements $m(v)$ and $m'(v)$ are equivalent (respectively, similar).

3. The Kuranishi Space

In this section we shall extend the Kuranishi's theory of locally complete families of complex structures ([Ku1], [Ku2]) to arbitrary elliptic differential graded Lie algebras. For finite dimensional differential graded Lie algebra this theory was developed by Nijenhouis and Richardson ([NR1], [NR2]). In [GM2] the formal analog of this theory was developed for an analytic differential graded Lie algebras.

3.1. The *Kuranishi* map $F : \widehat{L}^1 \to \widehat{L}^1$ is a quadratic map defined by

$$F(a) = a + \frac{1}{2}\delta[a, a].$$

We observe that $F(L^1) \subset L^1$. It is shown in [GM2] (Lemma 2.2) that there exist balls B and B' around 0 in $\widehat{L}^1$ such that F is an analytic diffeomorphism $B \to B'$. The *Kuranishi space* $\mathcal{K} \subset \mathcal{H}^1$ is defined by

$$\mathcal{K} = \Big\{ a \in B' \cap \mathcal{H}^1 : \ H\big[F^{-1}(a), F^{-1}(a)\big] = 0 \Big\}.$$

Let

$$Y = \big\{ a \in \mathcal{H}^1 \oplus \widehat{A}^1 : da + \frac{1}{2}[a, a] = 0 \big\}.$$

The following theorem is proven in [GM2] (Theorem 2.3)

3.2. Theorem. *F induces a homeomorphism from a neighborhood of 0 in Y to a neighborhood of 0 in $\mathcal{K}$.*

Since $\mathcal{K} \subset \mathcal{H}^1 \subset L^1$ and $F(L^1) \subset L^1$ it follows from this theorem that $Y \subset L^1$.

3.3. We shall say that the set $X \subset M$ is a *locally complete set of deformations* if there exists a neighborhood $U \subset M$ of the origin such that any $m \in U$ is equivalent to some element of X. Kuranishi ([Ku1],[Ku2]) showed that in the case of the Kodaira-Spencer algebra the set Y is locally complete. We shall show that it is also true for an elliptic differential graded Lie algebra. In fact, we need a bit stronger result.

3.4. Lemma. *Let (L, d) be an elliptic differential graded Lie algebra. Then there exist a neighborhood U of 0 in M, a neighborhood V of 0 in $\widehat{L}^0$ and a neighborhood W of 0 in $\widehat{A}^0$ such that for any $m \in U$ and any $s \in V$ there exists a unique element $f \in W$ such that*

$$\rho(s) \circ \rho(f)\,(m) \in Y.$$

Proof. Let $\Phi : \widehat{L}^1 \times \widehat{L}^0 \times \widehat{L}^0 \to \widehat{L}^1$ be the map defined by the formula:

$$\Phi(x, a, s) = \rho(s) \circ \rho(a)\,(x) - x + da + \left[\int_0^1 \exp((1-t)\,\mathrm{ad}\, s)\, dt\right] ds, \tag{3.1}$$

where $x \in L^1$ and $a \in L^0$. We need to find f and m' such that

$$\begin{gathered} m' = \rho(s) \circ \rho(f)\,(m), \\ f \in \widehat{A}^0, \qquad m' \in \mathcal{H}^1 \oplus \widehat{A}^1. \end{gathered} \tag{3.2}$$

Denote $l = df + m'$. Then (3.2) is equivalent to

$$\begin{gathered} l = m - \left[\int_0^1 \exp((1-t)\,\mathrm{ad}\, s)\, dt\right] ds + \Phi(m, \delta l, s), \\ m' = (H + \delta d) l, \qquad f = \delta l \end{gathered} \tag{3.3}$$

It follows from (2.3) and (1.5) that there exist a neighborhood U of 0 in M, a neighborhood V of 0 in $\widehat{L}^0$ and a neighborhood W of 0 in $\widehat{A}^0$ such that for any $m \in U$, any $s \in V$ and any $l, l' \in W$

$$m - \left[\int_0^1 \exp((1-t)\operatorname{ad} s)\, dt\right] ds + \Phi(m, \delta l, s) \in W$$

and

$$|\Phi(m, \delta l, s) - \Phi(m, \delta l', s)|_1 < \frac{1}{2}|l - l'|_1.$$

(Remind that by $|\cdot|_1$ we denote the norm on L^1). Then for any $m \in U, s \in V$ the equation (3.3) has a unique solution $l \in W$ given, say, by iterations

$$l^{(r+1)} = m - \left[\int_0^1 \exp((1-t)\operatorname{ad} s)\, dt\right] ds + \Phi(m, \delta l^{(r)}, s),$$
$$l^{(1)} = m - \left[\int_0^1 \exp((1-t)\operatorname{ad} s)\, dt\right] ds.$$

The lemma is proved. □

3.5. An analytic deformation $m : U \to M$ $(0 \in U \subset \mathbb{R}^n)$ of an elliptic differential graded Lie algebra is called a *Kuranishi deformation* if $m(u) \in Y$ for any $u \in U$.

Note that the element $f \in \widehat{L}^0$ defined by (3.3) depends analytically on m. Hence, we obtain the following

3.6. Corollary. *Let $m : U \to M$ $(0 \in U \subset \mathbb{R}^n)$ be an analytic deformation of an elliptic differential graded Lie algebra (L, d) and let $s : U \to L^0$ be an analytic map. Then there exist a neighborhood $V \subset U$ of 0 in $\mathbb{R}^n$ and a unique analytic function $f : V \to \widehat{A}^0$ such that $f(0) = 0$ and*

$$\rho(s(u)) \circ \rho(f(u))\, (m(u)) \in Y, \qquad u \in V.$$

Setting in the above lemmas $s \equiv 0$ we obtain the following:

3.7. Proposition. *1. Let (L, d) be an elliptic differential graded Lie algebra. Then there exists a neighborhood U of 0 in M such that any element $m \in M$ is similar to a unique $m' \in Y$. In particular, $Y = \left\{a \in \mathcal{H}^1 \oplus \widehat{A}^1 : da + \frac{1}{2}[a, a] = 0\right\}$ is a locally complete set of deformations.*

2. Any analytic deformation of an elliptic differential graded Lie algebra is similar to a unique Kuranishi deformation.

4. Formal Theory

In this section we shall show that for some deformation problems the existence of a formal power series solution guarantees the existence of an analytic solution.

4.1. By a (1-parameter) *formal deformation* of a differential graded Lie algebra (L, d) we shall understand a formal power series

$$m(t) = tm_1 + t^2 m_2 + \cdots$$

with coefficients m_n $(n = 1, 2, \dots)$ in L^1 satisfying

$$dm(t) + \frac{1}{2}[m(t), m(t)] = 0. \tag{4.1}$$

Note that any analytic deformation may be considered as a formal one.

Two formal deformations $m(t)$ and $m'(t)$ are called *formally equivalent* if there exists a formal power series

$$f(t) = tf_1 + t^2 f_2 + \cdots$$

with coefficients f_n $(n = 1, 2, \dots)$ in $\widehat{L}^0$ such that (cf. (2.4))

$$m'(t) = \rho(f(t))\, m(t) \stackrel{\text{def}}{=} \exp(\operatorname{ad} f(t))\, m(t) + \frac{I - \exp(\operatorname{ad} f(t))}{\operatorname{ad} f(t)}(df(t)). \tag{4.2}$$

If $f(t)$ in (4.2) may be chosen so that

$$f_n \in \widehat{A}^0 \tag{4.3}$$

for any $n = 1, 2, \dots,$ we shall say that the deformations $m(t)$ and $m'(t)$ are *formally similar*.

A formal deformation $m(t) = tm_1 + t^2 m_2 + \cdots$ is called a *formal Kuranishi deformation* if

$$m'_n \in \mathcal{H}^1 \oplus \widehat{A}^1 \tag{4.4}$$

for any $n = 1, 2, \dots.$

4.2. Lemma. *Let $m(t) = tm_1 + t^2 m_2 + \cdots$ be a formal deformation of an elliptic differential graded Lie algebra (L, d) and let $s(t) = ts_1 + t^2 s_2 + \cdots$ be a power series with coefficients in $\widehat{L}^0$. Then there exist a unique formal Kuranishi deformation $m'(t) = tm'_1 + t^2 m'_2 + \cdots$ and a unique power series $f(t) = tf_1 + t^2 f_2 + \cdots$ with coefficients in $\widehat{A}^0$ such that*

$$m'(t) = \rho(s(t)) \circ \rho(f(t))\,(m(t)). \tag{4.5}$$

Proof. Let $\Psi : \widehat{L}^1 \times \widehat{L}^0 \times \widehat{L}^0 \to \widehat{L}^1$ be the map defined by the formula

$$\Psi(x, a, s) = \rho(s) \circ \rho(a)(x) - x + da + ds.$$

The formal power series $f(t)$ and $m'(t)$ satisfy (4.3),(4.4) and (4.5) if and only if the formal power series $l(t) = tl_1 + t^2 l_2 + \cdots = df(t) + m'(t)$ satisfies the equation:

$$l(t) = m(t) - ds(t) + \Psi(m(t), \delta l(t), s(t)). \tag{4.6}$$

As the formal power series $\Psi(m(t), \delta l(t), s(t))$ does not contain summands of degree < 2 in t, the equation (4.6) defines the coefficients l_n of $l(t)$ recursively

$$l_n = l_n(m_1, \dots, m_n; l_1, \dots, l_{n-1}; s_1, \dots, s_n), \qquad n = 1, 2, \dots$$

Hence, there exists a unique formal solution $l(t)$ of (4.5). $\square$

The following corollary is the formal analogue of proposition 3.7.

4.3. Corollary. *Let $m(t) = m_1 + t^2 m_2 + \cdots$ be a formal deformation of an elliptic differential graded Lie algebra (L, d). Then there exists a unique formal Kuranishi deformation $m'(t) = m'_1 + t^2 m'_2 + \cdots$ which is formally similar to $m(t)$.*

The above lemma together with corollary 3.6 implies the following

4.4. Theorem. *If two analytic deformations of an elliptic differential graded Lie algebra are formally similar, then they are similar.*

Proof. Let analytic deformations $m'(t)$ and $m''(t)$ be formally similar and let $f(t) = tf_1 + t^2 f_2 + \cdots$ be the formal power series with coefficients in $\widehat{A}^0$ such that $m'(t) = \rho(f(t))\,(m''(t))$. We shall show that the power series $f(t)$ converges in a neighborhood of 0.

By proposition 3.7 there exists $\varepsilon > 0$ and an analytic function $s(t) \in \widehat{L}^0$ defined for $t \in (-\varepsilon, \varepsilon)$ such that $s(0) = 0$ and

$$\rho(s(t))\,(m'(t)) \in Y, \qquad t \in (-\varepsilon, \varepsilon).$$

Then

$$\rho(s(t)) \circ \rho(f(t))\,\big(m''(t)\big)$$

is a formal Kuranishi deformation.

By corollary 3.6 there exists $0 < \delta < \varepsilon$ and an analytic function $\widetilde{f}(t) \in \widehat{A}^0$ defined for $t \in (-\delta, \delta)$ such that

$$\rho(s(t)) \circ \rho(\widetilde{f}(t))\,\big(m''(t)\big) \in Y, \qquad t \in (-\delta, \delta).$$

The uniqueness statement in lemma 4.2 implies $f(t) \equiv \widetilde{f}(t)$. □

4.5. Theorem. *For any formal deformation $m(t)$ of an elliptic differential graded Lie algebra (L, d) and any integer n there exists an analytic deformation $\widetilde{m}(t)$ such that*

$$\widetilde{m}(t) \equiv m(t) \qquad \mod\ t^n. \tag{4.7}$$

Proof. Let $m'(t)$ be the unique formal Kuranishi deformation equivalent to $m(t)$ and let $f(t) = tf_1 + t^2 f_2 + \cdots$ be a formal power series with coefficients $f_n \in \widehat{L}^0$ such that

$$\rho(f(t))\,m(t) = m'(t).$$

Denote

$$m'_*(t) = tHm'_1 + t^2 Hm'_2 + \cdots .$$

Then $m'_*(t)$ is a formal power series with coefficients in the finite dimensional space $\mathcal{H}^1$. By theorem 3.2 the series $m'(t)$ is a formal Kuranishi deformation if and only if m'_* satisfies the system of analytic equations

$$H[F^{-1}(m'_*), F^{-1}(m'_*)] = 0. \tag{4.8}$$

By Artin's theorem [A] there exists an analytic solution $m_*(t) \in \mathcal{H}^1$ of (4.8) such that

$$m_*(t) \equiv m'_*(t) \qquad \mod\ t^n.$$

Denote $m''(t) = F^{-1}(m_*(t))$. Then $m''(t)$ is an analytic Kuranishi deformation. The deformation

$$\widetilde{m}(t) = \rho(-tf_1 - t^2 f_2 - \cdots - t^{n-1} f_{n-1})\,(m''(t))$$

is an analytic deformation and satisfies (4.7). □

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